Lecture 9 - Hidden Markov Models

Gidon Rosalki

2025-11-16

1 Definitions reminder

$$\mathbb{P}\left[X_1 \dots X_n, H_1 \dots H_n\right] = \mathbb{P}_0\left[H_0\right] \cdot \prod_i \mathbb{P}\left[H_{i+1} | H_i\right] \cdot \prod_i \mathbb{P}\left[X_i | H_i\right]$$

$$N_{0,s} = \sum_m \mathbb{1}_{H_1[m]=s}$$

$$N_{s,t} = \sum_m \sum_i \mathbb{1}_{H_{i-1}[m]=t, H_i[m]=s}$$

$$N_{s,x} = \sum_r \sum_i \mathbb{1}_{h_i[r]=s, X_i[r]=x}$$

2 Max-Max

Let there be

$$\Theta = \langle \mathbb{P}_0, \tau, \pi \rangle$$

The Data variable is simply a matrix of n by m, with each cell containing $X_{i_j}\left[j\right]$. We have max step where

$$\vec{H} \leftarrow arg \max_H \left\{ \vec{H} | Data \right\}$$

$$N^j \leftarrow \text{Count on } \vec{H}^j$$

And then we can get Max-Max $\max \mathbb{P}\left[H, X_i, \Theta\right]$

We will overall ask (a lot happened, I got lost)

$$\vec{N}^j \leftarrow \mathbb{E}\left[\vec{N} | \vec{X}, \Theta^j\right]$$

$$= \sum_{\vec{H}} \mathbb{P}\left[H | \vec{X}, \Theta^j\right] \vec{N}\left(H\right)$$

Here $\vec{N}$ is the collection of statistic defined earlier, concerning N.

$$\begin{aligned}
\mathbb{E}\left[\vec{N}|\vec{X},\Theta^j\right] &= \sum_m \mathbb{E}\left[\mathbb{1}_{H_1[m]=s}|\vec{X},\vec{\Theta}\right] \\
&= \sum_m \mathbb{P}\left[H_1\left[m\right]=s|\vec{X},\vec{\Theta}\right] \\
&= \sum_m \mathbb{P}\left[H_1\left[m\right]=s|\vec{X}\left[m\right],\vec{\Theta}\right] \\
&= \sum_m \frac{F_1^m\left[s\right]\cdot B_1^s\left[s\right]}{\mathbb{P}\left[\vec{X}\left[m\right]|\Theta\right]} \\
\mathbb{E}\left[N_{s,x}|\vec{X},\vec{\Theta}\right] &= \sum_m\sum_i \mathbb{E}\left[\mathbb{1}_{X_i=x,H_i=s}|\vec{X},\vec{\Theta}\right] \\
&= \sum_m\sum_i \mathbb{P}\left[X_i\left[m\right]=x,H_i\left[m\right]=s|\vec{X},\vec{\Theta}\right] \\
&= \sum_m\sum_{i,X_i[m]=x} \mathbb{P}\left[H_i\left[m\right]=s|\vec{X}\left[m\right],\vec{\Theta}\right] \\
&= \sum_m\sum_{i,X_i[m]=x} \frac{F_i^m\left[s\right]B_i^m\left[s\right]}{\mathbb{P}\left[\vec{X}\left[m\right]\right]} \\
\mathbb{E}\left[N_{s,t}|\vec{X},\vec{\Theta}\right] &= \sum_m\sum_{i=1}^{n_m-1} \mathbb{1}_{H_i[m]=s,H_{i+1}[m]=t|\vec{X},\vec{\Theta}} \\
&= \sum_m\sum_{i=1}^{n_m-1} \mathbb{P}\left[H_i\left[m\right]=s,H_{i+1}\left[m\right]=t|\vec{X}\left[m\right],\vec{\Theta}\right] \\
\mathbb{P}\left[H_i=s,H_{i+1}=t,\vec{X}\right] &= \mathbb{P}\left[X_1\dots X_i,H_i=s\right] \\
&\mathbb{P}\left[X_{i+2}\dots X_n|H_{i+1}=t,X_1\dots X_i,H_i=s\right] \\
&\mathbb{P}\left[H_{i+1}=t|H_i=s,X_1\dots X_i\right] \\
&\mathbb{P}\left[X_{i+1}|H_{i+1}=t,H_i=s,X_1\dots X_i,X_{i+2}\dots X_n\right] \\
&= \mathbb{P}\left[X_1\dots X_i,H_i=s\right] \\
&\mathbb{P}\left[X_{i+2}\dots X_n|H_{i+1}=t\right] \\
&\mathbb{P}\left[H_{i+1}=t|H_i=s\right] \\
&\mathbb{P}\left[X_{i+1}|H_{i+1}=t\right] \\
&= F_i\left[\right]B_{in}\left[t\right]\cdot\tau\left[t,s\right]\cdot\pi\left[X_{i+1},t\right] \\
\mathbb{E}\left[N_{s,t}|\vec{X},\vec{\Theta}\right] &= \sum_m\sum_{i=1}^{n_m-1} \mathbb{1}_{H_i[m]=s,H_{i+1}[m]=t|\vec{X},\vec{\Theta}} \\
&= \sum_m\sum_{i=1}^{n_m-1} \mathbb{P}\left[H_i\left[m\right]=s,H_{i+1}\left[m\right]=t|\vec{X}\left[m\right],\vec{\Theta}\right] \\
&= \sum_m\sum_i \frac{F_i^m\left[s\right]B_{i+1}^m\left[t\right]\tau\left[t,s\right]\pi\left[X_{i+1},t\right]}{\mathbb{P}\left[\vec{X}\left[m\right]\right]}
\end{aligned}$$

So let us write some pseudocode:

1

Input: *input*
Output: *output*
$\Theta^1 \leftarrow$ unit
for j=1... **do**
    Run forward / backward for all m
    Collect $\mathbb{E}\left[N^j\right]$ with F / B
    $\Theta^{j+1} \leftarrow MLE\left(\vec{N}^j\right)$
end for
return *state*